Chapter 5.2
Riemann Sums

In order to get the exact area under the curve \( y = f(x) \) from \( x = a \) to \( x = b \) we split the curve into \( n \) equal width rectangles, this width of one such rectangle is called \( \Delta x \) (read as delta \( x \)) and its value is \( \frac{b-a}{n} \). We then find the height of each rectangle, in this case the height of a typical rectangle which will be the value of the function at the point \( x_i^* \) i.e. \( f(x_i^*) \).

We then find the area of this typical rectangle by multiplying its width and height together to get \( f(x_i^*) \Delta x \). We then add all these rectangles together to get an estimate for the complete area under the curve, this is written using sigma notation as \( \sum_{i=1}^{n} f(x_i) \Delta x \). We then assume that we let the number of strips \( n \) become infinite so that the exact area under the curve will be

\[
\text{Exact Area under the curve} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]

This process essentially says that we can get the exact area under a curve called \( \int_{a}^{b} f(x) \, dx \) (read as the integral of \( f(x) \) from \( x = a \) to \( x = b \)) by splitting the area into an infinite number of rectangular strips and adding their areas together.

This process is stated more formally by the theorem given below.

**Definition of a Definite Integral** If \( f \) is a function defined for \( a \leq x \leq b \), we divide the interval \([a, b]\) into \( n \) subintervals of equal width \( \Delta x = (b - a)/n \). We let \( x_0 = a, x_1, x_2, \ldots, x_n = b \) be the endpoints of these subintervals and we let \( x_1^*, x_2^*, \ldots, x_n^* \) be any sample points in these subintervals, so \( x_i^* \) lies in the \( i \)th subinterval \([x_{i-1}, x_i]\). Then the definite integral of \( f \) from \( a \) to \( b \) is

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x
\]

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that \( f \) is integrable on \([a, b]\).
If the function $f(x)$ is continuous then we can state the previous theorem in the concise form as

**4 Theorem**  If $f$ is integrable on $[a, b]$, then

$$
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i) \Delta x
$$

where

$$
\Delta x = \frac{b - a}{n} \quad \text{and} \quad x_i = a + i \Delta x
$$

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**Example 1:** Find estimates for the area under the curve of $f(x) = x^2$ from $x = 2$ to $x = 10$ using 4 strips.

$$
\Delta x = \frac{b-a}{n} = \frac{10-2}{4} = 2 \quad \text{and} \quad x_i = a + i \Delta x = 2 + i(2) = 2 + 2i
$$

So we have the following $x$-values

<table>
<thead>
<tr>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

| $f(x)$ | 4     | 16    | 36    | 64    | 100   |

$$
R_4 = \sum_{i=1}^4 f(x_i) \Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x = 16(2) + 36(2) + 64(2) + 100(2) = 332
$$

$$
L_4 = \sum_{i=0}^3 f(x_i)\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x = 4(2) + 16(2) + 36(2) + 64(2) = 240
$$

To find an estimate for the area under this curve using mid-points we need to create a new table using the midpoint of each interval.

For example the first mid-point has an $x$-coordinate of $\bar{x}_1 = \frac{x_0 + x_1}{2} = \frac{2+4}{2} = 3$

<table>
<thead>
<tr>
<th>$\bar{x}_1$</th>
<th>$\bar{x}_2$</th>
<th>$\bar{x}_3$</th>
<th>$\bar{x}_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
</tr>
</tbody>
</table>

| $f(x)$ | 9     | 25    | 49    | 81    |

$$
M_4 = \sum_{i=1}^4 f(\bar{x}_i)\Delta x = f(\bar{x}_1)\Delta x + f(\bar{x}_2)\Delta x + f(\bar{x}_3)\Delta x + f(\bar{x}_4)\Delta x = 9(2) + 25(2) + 49(2) + 81(2) = 328
$$
Example 2: Find estimates for the area under the curve of \( f(x) = x^3 - 6x \) from \( x = 0 \) to \( x = 3 \) using the right endpoints for each rectangle. If we use

In this example if we let \( n = 40 \) strips then \( \Delta x = \frac{b-a}{n} = \frac{3-0}{40} = \frac{3}{40} \)

\[
x_i = a + i\Delta x = 0 + i \frac{3}{40}
\]

The Reimann Sum for 40 strips would be

\[
R_{40} = \sum_{i=1}^{40} f(x_i)\Delta x
\]

\[
= \sum_{i=1}^{40} f \left( i \frac{3}{40} \right) \frac{3}{40}
\]

\[
= \sum_{i=1}^{40} \left( \left( i \frac{3}{40} \right)^3 - 6 \left( i \frac{3}{40} \right) \right) \frac{3}{40}
\]

This gives us an estimate of the area using 40 strips to be \( R_{40} = -6.3998 \)

If you then increase the number of strips used to \( n = 100, 500, 1000 \) and then 5000 you will get progressively more accurate estimates as shown in the table above.

**Note:** The area under this curve is a negative value, since the majority of the curve lies below the x-axis. In this particular example we can be sure that the correct answer is \( 0 - 6.7 \) to one decimal place which is not that accurate considering we used as many as 5000 strips!
Example 3: Use Riemann Sums to show that the area \( \int_a^b x \, dx = \frac{1}{2} (b^2 - a^2) \)

Solution: We start by finding \( \Delta x = \frac{b-a}{n} \) and \( x_i = a + i \Delta x = a + \frac{(b-a)}{n} i \)

Next we write down the definition of the Riemann sum for this situation.

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i) \Delta x
\]

\[
\int_a^b x \, dx = \lim_{n \to \infty} \sum_{i=1}^n f \left( a + \frac{(b-a)}{n} i \right) \frac{b-a}{n}
\]

\[
\int_a^b x \, dx = \lim_{n \to \infty} \sum_{i=1}^n \left( a + \frac{(b-a)}{n} i \right) \frac{b-a}{n}
\]

\[
\int_a^b x \, dx = \lim_{n \to \infty} \sum_{i=1}^n \frac{a(b-a)}{n} + \frac{(b-a)^2}{n^2} i
\]

\[
\int_a^b x \, dx = \lim_{n \to \infty} \left( \frac{a(b-a)}{n} \sum_{i=1}^n 1 + \frac{(b-a)^2}{n^2} \sum_{i=1}^n i \right)
\]

\[
\int_a^b x \, dx = \lim_{n \to \infty} \frac{a(b-a)}{n} n + \frac{(b-a)^2}{n^2} \frac{n(n+1)}{2}
\]

\[
\int_a^b x \, dx = \lim_{n \to \infty} a(b-a) + (b-a)^2 \left( \frac{n+1}{2n} \right)
\]

\[
\int_a^b x \, dx = \lim_{n \to \infty} a(b-a) + (b-a)^2 \left( \frac{\frac{1}{2} + \frac{1}{2n}}{2n} \right)
\]

\[
\int_a^b x \, dx = a(b-a) + (b-a)^2 \left( \frac{1}{2} + 0 \right)
\]

\[
\int_a^b x \, dx = ab - a^2 + \frac{1}{2} (b-a)^2
\]

\[
\int_a^b x \, dx = ab - a^2 + \frac{1}{2} (b^2 - 2ab + a^2)
\]

\[
\int_a^b x \, dx = ab - a^2 + \frac{1}{2} b^2 - ab + \frac{1}{2} a^2
\]

\[
\int_a^b x \, dx = \frac{1}{2} b^2 - \frac{1}{2} a^2
\]

\[
\int_a^b x \, dx = \frac{1}{2} (b^2 - a^2)
\]
Example 4: Use Riemann Sums to show that the area $\int_a^b x^2 \, dx = \frac{1}{3} (b^3 - a^3)$

Solution: We start by finding $\Delta x = \frac{b-a}{n}$ and $x_i = a + i \Delta x = a + \frac{(b-a)}{n} i$

Next we write down the definition of the Riemann sum for this situation.

\[
\int_a^b f(x) \, dx = \sum_{i=1}^{n} f(x_i) \Delta x
\]

\[
\int_a^b x^2 \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(a + \frac{(b-a)}{n} i\right) \frac{b-a}{n}
\]

\[
\int_a^b x^2 \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left(a^2 + \frac{2a(b-a)}{n} i + \frac{(b-a)^2}{n^2} i^2\right) \frac{b-a}{n}
\]

\[
\int_a^b x^2 \, dx = \lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{(b-a)a^2}{n} + \sum_{i=1}^{n} \frac{2a(b-a)^2}{n^2} i + \sum_{i=1}^{n} \frac{(b-a)^3}{n^3} i^2\right)
\]

\[
\int_a^b x^2 \, dx = \lim_{n \to \infty} \left(\frac{(b-a)a^2}{n} \sum_{i=1}^{n} 1 + \frac{2a(b-a)^2}{n^2} \sum_{i=1}^{n} i + \frac{(b-a)^3}{n^3} \sum_{i=1}^{n} i^2\right)
\]

\[
\int_a^b x^2 \, dx = \lim_{n \to \infty} \left(\frac{(b-a)a^2}{n} \frac{n(n+1)}{2} + \frac{2a(b-a)^2}{n^2} \frac{n(n+1)(2n+1)}{6} + \frac{(b-a)^3}{n^3} \frac{2n^2+3n+1}{6n^2}\right)
\]

\[
\int_a^b x^2 \, dx = \lim_{n \to \infty} \left((b-a)a^2 + a(b-a)^2 \left(1 + \frac{1}{n}\right) + (b-a)^3 \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)\right)
\]

\[
\int_a^b x^2 \, dx = (b-a)a^2 + a(b-a)^2 (1+0) + (b-a)^3 \left(\frac{1}{3} + 0 + 0\right)
\]

\[
\int_a^b x^2 \, dx = (b-a)a^2 + a(b-a)^2 + \frac{1}{3} (b-a)^3
\]

\[
\int_a^b x^2 \, dx = (ba^2 - a^3) + a(b^2 - 2ab + a^2) + \frac{1}{3} (b^3 - 3ab^2 + 3a^2b - a^3)
\]

\[
\int_a^b x^2 \, dx = ba^2 - a^3 + ab^2 - 2a^2b + a^3 + \frac{1}{3} b^3 - ab^2 + a^2b - \frac{1}{3} a^3
\]

\[
\int_a^b x^2 \, dx = \frac{1}{3} b^3 - \frac{1}{3} a^3
\]

\[
\int_a^b x^2 \, dx = \frac{1}{3} (b^3 - a^3)
\]